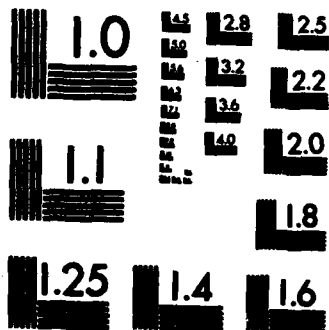|	1.0			2.8	2.5
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MRC Technical Summary Report #2693

A STUDY OF THE DRAWING PROCESS FOR
OPTICAL FIBERS

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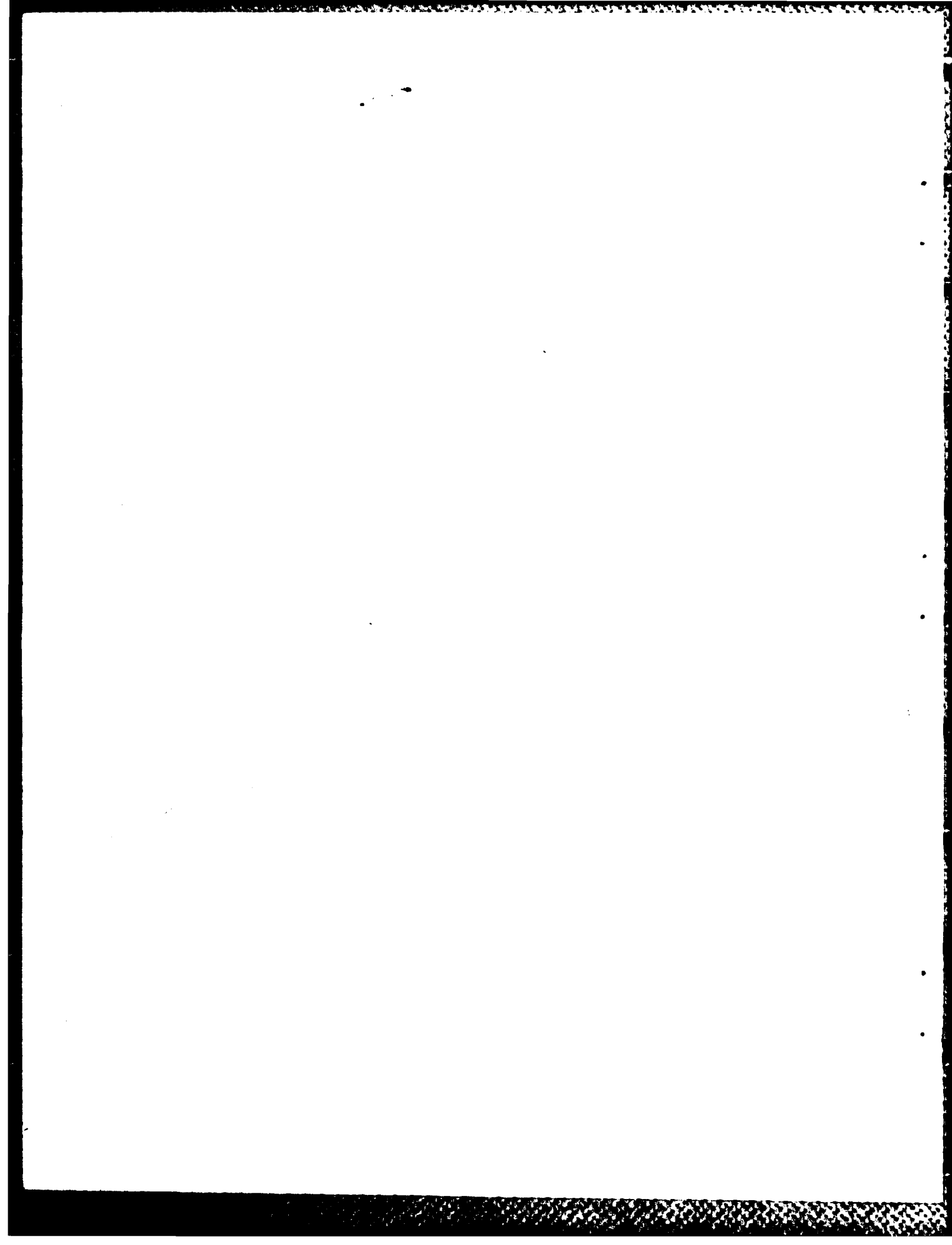May 1984

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A STUDY OF THE DRAWING PROCESS FOR OPTICAL FIBERS

John C. Strikwerda

Technical Summary Report #2693
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ABSTRACT

Optical fibers are produced by heating a silica rod and pulling on the
rod to produce a thin fiber. This process involves several interacting
effects, especially the temperature dependent viscosity of the rod, and the
shape of the rod in the transition zone. In this paper the equations
describing this process are studied by means of a perturbation analysis using
as the perturbation parameter the ratio of the rod to the length of the
furnace. The silica rod is usually doped so that it will have special optical
properties and it is important to know the concentration of the dopant in the
final fiber. Simple equations for the significant terms in the perturbation
series are obtained for all quantities of interest. A simple formula is
obtained which expresses the radius of the fiber in terms of the initial
radius of the rod, the drawing force and other important parameters.

SIGNIFICANCE AND EXPLANATION

Optical fibers are important as a new means of communication. In this report the equations which describe the production process for the fibers are studied. Through a perturbation analysis relatively simple equations are obtained which describe the process.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A STUDY OF THE DRAWING PROCESS FOR OPTICAL FIBERS

John C. Strikwerda

1. Introduction

Optical fibers are usually produced from silica rods by heating and pulling the rod in the axial direction. This essentially simple physical process is difficult to analyze mathematically due to the interaction of several effects. These effects include the temperature of the furnace, the temperature dependent viscosity of the rod, the drawing or pulling force on the rod, and the shape of the rod during the drawing process. In addition the silica rod is usually doped so that it will have special optical properties and it is important to know the radial dependence of the dopant in the final fiber. In this report this process is studied so as to produce convenient equations from which to calculate many of these quantities and to demonstrate the essential features of this process. In a subsequent report these equations will be solved numerically to produce quantitative results.

2. Description of the Problem

Consider a circular rod placed in a furnace, which we assume as cylindrical in shape, with temperature distribution in the furnace given by $T_f(Z)$. The rod and furnace are taken as coaxial with the Z-axis, and for simplicity of analysis both the rod and furnace are assumed to be infinite in extent. (In general, the furnace will be cold for all but a finite interval.) The rod is forced to move in the axial direction under the influence of a given drawing force F_d. The initial feed rate of the rod is U_o and the radius is R_o. Because of the temperatures of the furnace the rod will behave as a viscous liquid and be drawn down to a radius R_1 and have velocity U_1. An objective of this study is

to show how the temperature distribution $T_f(Z)$, the drawing force F_d, and feed rate U_o affect the radius reduction from R_o to R_1. The rod and furnace are assumed to have no angular variation, which means all effects will depend only on the axial and radial coordinates. Paek and Runk (1978) presented a method for determining the neck-down shape of the rod, however there are several difficulties with the equations which they employed. The equations should allow for the degenerate case in which the rod and furnace are at the same constant temperature. The equations of Paek and Runk do not have such a solution unless the temperature is 1100°C, which appears to have been chosen quite arbitrarily. The equations used here do not have such arbitrariness. Because of the greater generality of the equations employed here they should be useful in studying the cooling of the fiber after it exits from the furnace. The cooling process is of interest because of its role in determining the strength of the fiber.

The method used in this analysis is the method of perturbation expansions using the ratio of the rod radius to a length representative of the heating zone. This slenderness ratio is of the order of 0.05 for the draw-down process.

The use of the slenderness ratio has been employed to study slender jets of inviscid fluids with great success, see the work of Geer (1977a,1977b) and has led to an effective numerical method, see Strikwerda and Geer (1980). (See also Geer and Strikwerda (1980) and (1983).) An advantage of the perturbation method is that it displays the relative importance of the various effects which interact in the process. For the draw-down process the perturbation method gives relatively simple formulas for the determination of the temperature, shape, and dopant concentration. An especially striking formula is that for the rod shape as a function of the viscosity of the rod.

3. The Governing Equations

The governing equations will be given first in dimensional form and then rewritten in nondimensional form. The convention which is adopted is that for quantities denoted by Latin letters the dimensional quantity will be denoted by the upper case and the

nondimensional quantity by the lower case. The exception is temperature for which Θ will denote the nondimensional temperature. For quantities denoted by Greek letters an overbar will distinquish the dimensional value. The exception here is that $\bar{\epsilon}$ denotes the emissivity but ϵ is the slenderness ratio of the perturbation expansion.

The equations describing the drawing process are essentially those given by Paek and Runk (1978). The differences between the equations sets are that in this paper the pressure is included in the stress, the heat flux is calculated differently, and here the dopant concentration equation is also included.

The equations governing the movement of the rod are the balance of forces

$$\frac{\partial \bar{\sigma}_{zz}}{\partial z} + \frac{1}{R}\frac{\partial(R\bar{\sigma}_{rr})}{\partial R} = 0$$

$$\frac{\partial \bar{\sigma}_{rz}}{\partial z} + \frac{1}{R}\frac{\partial(R\bar{\sigma}_{rr})}{\partial R} - \frac{\bar{\sigma}_{\phi\phi}}{R} = 0$$

where the $\bar{\sigma}_{ij}$ are the usual components of the stress tensor. At the surface of the rod we have the conditions of no stress

$$\bar{\sigma}_{zz}\, n_z + \bar{\sigma}_{rz}\, n_r = 0$$

$$\bar{\sigma}_{rz}\, n_z + \bar{\sigma}_{rr}\, n_r = 0$$

where (n_z, n_r) are the coordinates of the unit normal at the surface. Surface tension is ignored as it is negligible for silica rods (Geyling and Homsy 1980)).

The rod is an incompressible medium so the velocity components satisfy

$$\frac{\partial U}{\partial z} + \frac{1}{R}\frac{\partial Rv}{\partial R} = 0$$

The liquid phase is assumed to be a Newtonian fluid (see Manfre (1969)) so the stress components in the liquid are given by

$$\bar{\sigma}_{zz} = - P + 2\bar{\mu}\frac{\partial U}{\partial z}$$

$$\bar{\sigma}_{rz} = \bar{\mu}(\frac{\partial U}{\partial R} + \frac{\partial V}{\partial z})$$

$$\bar{\sigma}_{rr} = - P + 2\mu\frac{\partial V}{\partial R}$$

$$\bar{\sigma}_{\phi\phi} = - P + 2\mu\ V/R$$

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where P is the pressure and $\bar{\mu}$ is the viscosity. Gravitational effects can be included in the pressure, or ignored as we do here (Geyling and Homsy (1980)). The equation for the temperature is

$$\bar{\rho} \, C_v (U \frac{\partial T}{\partial Z} + V \frac{\partial T}{\partial R}) = \nabla^2 (K_c T + \frac{4n_o^2 \, \bar{\sigma}}{3 \, \bar{\gamma}} \, T^4)$$

in the rod and

$$\frac{\partial}{\partial n} (K_c \, T + \frac{4n_o^2 \, \bar{\sigma}}{3\bar{\gamma}} \, T^4) = - \, Q(Z)$$

on the surface, where $Q(Z)$ is the heat flux due to the furnace (Paek and Runk (1978)). $Q(Z)$ is given by

$$Q(Z) = \bar{\epsilon} \, \bar{\sigma} \int_{-\infty}^{\infty} \int_{-\phi_o(Z,\xi)}^{\phi_o(Z,\xi)} (T(Z)^4 - T_f(\xi)^4) \frac{\cos\phi_1 \cos\phi_2}{\pi \, R_{12}^2} \, Ad\phi d\xi$$

where R_{12} is the distance between the point P_1 on the rod surface at Z and the point P_2 on the furnace surface at ξ, (see Sparrow and Cess (1967)). The angle ϕ_1 is that between the normal at P_1 and the line connecting P_1 to P_2, and ϕ_2 is the angle between that line and the normal at P_2. The radius of the furnace is A and $\phi_o(Z,\xi)$ is the angle intercepted by the tangent plane at P_1 on the section of the furnace at ξ. The other quantitites in the above equations are the density $\bar{\rho}$, the specific heat C_v, the thermal conductivity K_c, the index of refraction n_o, the absorption coefficient $\bar{\gamma}$, the emissivity $\bar{\epsilon}$, and the Stefan-Boltzmann constant $\bar{\sigma}$.

The equation governing the transport of dopant is

$$U \frac{\partial C}{\partial Z} + V \frac{\partial C}{\partial R} = \frac{\partial}{\partial Z} (K(T) \frac{\partial C}{\partial Z}) + \frac{1}{R} \frac{\partial}{\partial R} (RK(T) \frac{\partial C}{\partial R})$$

where $C(R,Z)$ is the concentration of dopant and $K(T)$ is the diffusivity. On the surface the condition is

$$\frac{\partial C}{\partial n} = 0.$$

The functions that are presumed known are the viscosity $\bar{\mu}(T)$ and diffusivity $K(T)$, as functions of the temperature, and the temperature of the furnace wall $T_f(Z)$ as a function of the axial distance Z.

The variables are nondimensionalized by expressing them as multiples of various constant reference quantities. The reference velocity is the feed rate U_o, the reference force is the drawing force F_d, the reference temperature is the initial temperature of the rod T_o in an absolute scale, e.g. °K. The initial furnace temperature is assumed to be the same as that of the rod. There are two reference lengths, one being the initial rod diameter R_o, the other being a representative length of the furnace Z_o. The ratio R_o/Z_o is assumed to be small.

The nondimensional variables are defined by

$$\epsilon = R_o/Z_o \qquad\qquad r = R/R_o \qquad\qquad z = Z/Z_o$$

$$u = U/U_o \qquad\qquad v = V/(U_o \epsilon)$$

$$\theta = T/T_o \qquad\qquad k_o(\theta) = K(T)Z_o/U_o R_o^2$$

The reference stress is

$$\bar{\sigma}_o = F_d/R_o^2$$

and so

$$\sigma_{ij} = \bar{\sigma}_{ij}/\bar{\sigma}_o$$

$$\mu = \bar{\mu} \, U/(\bar{\sigma}_o Z_o)$$

$$p = P/\bar{\sigma}_o.$$

The concentration is normalized so that initially

$$\int_0^A C(R,Z)R\,dR = \tfrac{1}{2}A^2 C_o$$

and $c = C/C_o$.

The nondimensional equations which hold in the rod are

$$(1) \qquad \epsilon \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial r\sigma_{rz}}{r\partial r} = 0$$

$$(2) \qquad \epsilon \frac{\partial \sigma_{rz}}{\partial z} + \frac{\partial r\sigma_{rr}}{r\partial r} - \frac{\sigma_{\phi\phi}}{r} = 0$$

$$(3) \qquad \frac{\partial u}{\partial z} + \frac{\partial rv}{r\partial r} = 0$$

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$$(4) \qquad u\,\frac{\partial\theta}{\partial z} + v\,\frac{\partial\theta}{\partial r} = \left(\frac{1}{r}\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r} + \epsilon^2\,\frac{\partial^2}{\partial z^2}\right)(k_1\theta + k_2\theta^4)$$

$$(5) \qquad u\,\frac{\partial c}{\partial z} + v\,\frac{\partial c}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(rk_o(\theta)\,\frac{\partial c}{\partial r}\right) + \epsilon^2\,\frac{\partial}{\partial z}\left(k_o(\theta)\,\frac{\partial c}{\partial z}\right)$$

In the liquid phase of the rod the stresses are given by

$$(6) \qquad \sigma_{zz} = -p + 2\mu\,\frac{\partial u}{\partial z}$$

$$(7) \qquad \sigma_{rz} = \epsilon^{-1}\mu\left(\frac{\partial u}{\partial r} + \epsilon^2\,\frac{\partial v}{\partial z}\right)$$

$$(8) \qquad \sigma_{rr} = -p + 2\mu\,\frac{\partial v}{\partial r}$$

$$(9) \qquad \sigma_{\phi\phi} = -p + 2\mu\,v/r.$$

The normal to the rod surface is given by

$$n_z = \frac{-\epsilon s'(z)}{\sqrt{1+\epsilon^2 s'(z)^2}}\;,\quad n_r = \frac{1}{\sqrt{1+\epsilon^2 s'(z)^2}}$$

where the rod surface is given by $r = s(z)$.

The boundary conditions are therefore

$$(10) \qquad \sigma_{rz} = \epsilon\,s'(z)\,\sigma_{zz}$$

$$(11) \qquad \sigma_{rr} = \epsilon\,s'(z)\,\sigma_{rz}$$

$$(12) \qquad v = s'(z)\,u.$$

$$(13) \qquad \left(\frac{\partial}{\partial r} - \epsilon^2 s'\,\frac{\partial}{\partial z}\right)(k_1\theta + k_2\theta^4) = -q(z)(1+\epsilon^2 s'(z)^2)^{1/2}$$

$$(14) \qquad q(z) = \epsilon\,k_3\,a\int_{-\infty}^{\infty}\int_{-\phi_o(z,\xi)}^{\phi_o(z,\xi)}(\theta(z)^4 - \theta_f(\xi)^4)\,\frac{\cos\phi_1\cos\phi_2}{\pi\,r_{12}^2}\,d\phi\,d\xi$$

$$(15) \qquad \frac{\partial c}{\partial r} = \epsilon^2 s'\,\frac{\partial c}{\partial z}$$

Because of the nondimensionalization which is employed, the values of the several variables in the initial solid phase of the rod (i.e. at $z = -\infty$) are:

$$u(r,-\infty) = 1, \quad v(r,-\infty) = 0, \quad \theta(r,-\infty) = 1$$

$$\mu(1) = \infty, \quad k_o(1) = 0, \quad \sigma_{zz}(r-\infty) = 1$$

and the other stress components vanish,

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and we assume that $q(z)$ vanishes as $z \to -\infty$. Also $c(r,-\infty)$ is some specified function. The initial radius of the rod is 1.

If θ^* is the nondimensional softening temperature of the rod, then

$$\left.\begin{array}{l} \mu(\theta) = \infty \\[6pt] k_o(\theta) = 0 \end{array}\right\} \quad \text{if} \quad \theta < \theta^*$$

and

$$\left.\begin{array}{l} \mu(\theta) < \infty \\[6pt] k_o(\theta) > 0 \end{array}\right\} \quad \text{if} \quad \theta > \theta^* .$$

A possible expression for $\mu(\theta)$ for $\theta > \theta^*$ is $\mu(\theta) = \mu^* e^{b/(\theta - \theta^*)}$, see Napolitano and Hawkins (1964). The nondimensional constants are

$$k_1 = (K_c Z_o)/(\bar{\rho} C_v U_o R_o^2)$$

$$k_2 = (4 \bar{T}_o^3 n_o^2 \bar{\sigma} Z_o)/(3 \bar{\gamma} \rho C_v U_o R_o^2)$$

(16)

$$k_3 = (\bar{\epsilon \sigma} T_o^3)/(\bar{\rho} C_v U_o Z_o)$$

$$a = \lambda/R_o$$

4. Perturbation Analysis for the Shape

Each of the variables in the process are expanded in perturbation series. For example,

$$u(r,z) = u_o(r,z) + \epsilon u_1(r,z) + \epsilon^2 u_2(r,z) + \cdots ,$$

$$s(z) = s_o(z) + \epsilon s_1(z) + \epsilon^2 s_2(z) + \cdots ,$$

also

$$\mu(\theta(r,z)) = \mu_o(r,z) + \epsilon \mu_1(r,z) + \epsilon^2 \mu_2(r,z) + \cdots .$$

Because of the subscripts on the stresses the perturbation index for the stresses will be a superscript, and σ_{rz} will be the only variable with a index of -1, i.e.

$$\sigma_{rz} = \epsilon^{-1} \, \sigma_{rz}^{-1} + \epsilon^{0} \, \sigma_{rz}^{0} + \cdots \, .$$

From equation (1) it follows that

$$\frac{\partial r \sigma_{rz}^{-1}}{r \partial r} = \frac{\partial r \sigma_{rz}^{0}}{r \partial r} = 0,$$

thus $r\sigma_{rz}^{-1}$ and $r\sigma_{rz}^{0}$ are independent of r. The boundary condition (10) further shows that σ_{rz}^{-1} and σ_{rz}^{0} vanish identically. Since

$$\sigma_{rz}^{-1} = \mu_o \, \frac{\partial u_o}{\partial r} = 0$$

it follows that u_o is a function only of z. From the continuity equation (3) and (12) there follows

$$\frac{du_o(z)}{dz} \, \frac{s_o^2}{2} = \int_0^{s_o} \frac{\partial u_o}{\partial z} \, r \, dr = - \int_0^{s_o} \frac{\partial r v_o}{\partial r} \, dr$$

$$= - \, s_o(z) v_o = - \, s_o(z) \, s_o'(z) \, u_o(z).$$

Therefore, since $u_o(z) = 1$ in the initial solid phase,

(17) $$u_o(z) = 1/s_o(z)^2 \, .$$

From (3) and (12) it follows that

(18) $$v_o(r,z) = r \, s_o'(z)/s_o(z)^3.$$

Equation (1) integrated over a section, together with (10) yields

$$\int_0^{s_o(z)} \frac{\partial \sigma_{zz}^{0}}{\partial z} \, rdr = - \int_0^{s_o(z)} \frac{\partial r \sigma_{rz}^{1}}{\partial r} \, dr$$

$$= - \, s_o(z) \, \sigma_{rz}^{1} - = - \, s_o(z) \, s_o'(z) \, \sigma_{zz}^{0} \, ,$$

which can be rewritten as

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$$\frac{d}{dz} \left(\int_0^{s_o(z)} \sigma_{zz}^o \, r dr \right) = 0.$$

Based on the initial solid phase, it follows that

(19)
$$\int_0^{s_o(z)} \sigma_{zz}^o \, r dr = \frac{1}{2}$$

From equation (2), with $\sigma_{rz}^{-1} = 0$,

$$\frac{1}{r} \frac{\partial r \sigma_{rr}^o}{\partial r} - \frac{\sigma_{\phi\phi}^o}{r} = 0$$

and substituting (8) and (9) gives

$$\frac{\partial}{\partial r} \left(-p_o + 2 \frac{s_o'(z)}{s_o(z)^3} \, \mu_o \right) = \frac{\partial}{\partial r} \sigma_{rr}^o = 0.$$

Since (11) shows that σ_{rr}^o vanishes on the surface of the rod, the conclusion is that

$$\sigma_{rr}^o = 0$$

or

(20)
$$p_o(r,z) = 2 \frac{s_o'(z)}{s_o(z)^3} \, \mu_o(r,z).$$

By the definition of σ_{zz} in the liquid phase, with (17) and (20)

$$\sigma_{zz}^o = -p_o + \mu_o \frac{\partial u_o}{\partial z}$$

$$= -6 \mu_o(r,z) \, s_o'(z)/s_o(z)^3.$$

At locations at which the rod is entirely in the liquid phase, this formula with (19) gives

$$-6 s_o'(z)/s_o(z)^3 \int_0^{s_o(z)} \mu_o(r,z) \, r dr = \frac{1}{2}.$$

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In the intergral in r make the substitution

(21) $$r = s_o(z) \, \eta,$$

which will prove very useful later as well, to obtain

(22) $$\frac{s_o'(z)}{s_o(z)} = -\frac{1}{12} \left(\int_0^1 \mu_o(\eta,z)\eta d\eta \right)^{-1}.$$

At this point it is convenient to make an approximation by replacing $\mu_o(r,z)$ by $\bar{\mu}_o(z)$, which is independent of r. The value of $\bar{\mu}_o(z)$ can be obtained in any of several ways. With this approximation (22) can be simply integrated to give the remarkable formula

(23) $$s_o(z) = \exp\left(-\frac{1}{6} \int_{-\infty}^z \bar{\mu}(\zeta)^{-1} d\zeta\right).$$

It is important to point out that the validity of the approximation of μ_o by $\bar{\mu}$ can be checked by the results of computations. If the computations show that for a appreciable interval of z values there are substantial solid and liquid portions for each value of z then the above approximation is not valid. In that case, if the rod is solid for r less than $\alpha_o(z) \, s_o(z)$, equations (22) is replaced by

(24) $$\frac{s_o'}{s_o(1-(\alpha_o(z)s_o(z))^2)} = -\frac{1}{12} \left(\int_{\alpha_o(z)}^1 \mu_o(\eta,z)\eta d\eta \right)^{-1}$$

which follows directly from (19) and σ_{zz}^o equal to 1.0 in the solid phase.

5. The Effect of Draw-down on the Dopant Concentration.

The equation governing $c_o(r,z)$ is

$$u_o \frac{\partial c_o}{\partial z} + v_o \frac{\partial c_o}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \, k_o(\theta_o) \frac{\partial c_o}{\partial r} \right).$$

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It is advantageous to change coordinates from (r,z) to (η,z) where η is defined by (21). The resulting equation is

$$u_o \frac{\partial c_o}{\partial z} + (v_o - u_o \eta\, s_o') \frac{1}{s_o} \frac{\partial c_o}{\partial \eta}$$

$$= \frac{1}{s_o(z)^2}\; \frac{1}{\eta} \frac{\partial}{\partial \eta} (\eta\, k_o(\theta_o) \frac{\partial c_o}{\partial \eta})\; .$$

However, from (17) and (18)

$$v_o - u_o \eta\, s_o' = 0$$

and again by (17) the equation simplifies to

(25)
$$\frac{\partial c_o}{\partial z} = \frac{1}{\eta} \frac{\partial}{\partial \eta} (\eta\, k_o(\theta_o) \frac{\partial c_o}{\partial \eta}),\quad z > -\infty,\; 0 < \eta < 1.$$

which is a simple parabolic differential equation for c_o. The initial conditions are that $c_o(\eta,z)$ is specified in the solid portion of the rod, (i.e. where $\eta = r$). The boundary condition (15) becomes

(26)
$$\frac{\partial c_o}{\partial \eta} = 0 \qquad \text{at}\quad \eta = 1.$$

Thus, the change of c_o is not explicitly dependent on the rod shape, depending only on the temperature θ_o, which does depend on the shape. Equations (25) and (26) are very amenable to numerical computation since the η variable is on a fixed domain.

6. Determination of the Temperature.

The equation for the nondimensional temperature $\theta_o(r,z)$ is

$$u_o \frac{\partial \theta_o}{\partial z} + v_o \frac{\partial \theta_o}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial}{\partial r} (k_1\theta_o + k_2\theta_o^4))$$

Using the transformation (21) as in the previous section this becomes

(27)
$$\frac{\partial \theta_o}{\partial z} = \frac{1}{\eta} \frac{\partial}{\partial \eta} (\eta \frac{\partial}{\partial \eta} (k_1\theta_o + k_2\theta_o^4))$$

The boundary condition (13) becomes

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(28)
$$\frac{\partial}{\partial \eta} (k_1 \theta_o + k_2 \theta_o^4) = -s_o(z) \, q_o(z).$$

The determination of $q_o(z)$ requires some analysis to obtain the limit as ε tends to zero.

For convenience, set

$$F(z, \zeta, \varepsilon) = \theta^4(s(z), z) - \theta_f^4(\zeta)$$

then from (14)

(29)
$$q_o(z) = k_3 \, a \, \lim_{\varepsilon \to 0} \varepsilon \int_{-\infty}^{\infty} \int_{-\phi_o}^{\phi_o} F(z, \zeta) \, \frac{\cos\phi_1 \, \cos\phi_2}{\pi \, r_{12}^2} \, d\phi d\zeta.$$

The coordinates of the point P_1 on the rod surface at z can be taken as $(z, \varepsilon s(z), 0)$ in cartesian coordinates and those of the point P_2 on the furnace surface at ζ are $(\zeta, \varepsilon a \cos \phi, \varepsilon \, a \, \sin \phi)$. Thus

$$r_{12}^2 = (z-\zeta)^2 + \varepsilon^2(s-a \cos \phi)^2 + \varepsilon^2(a \sin \phi)^2$$
$$= (z-\zeta)^2 + \varepsilon^2(s^2 + a^2 - 2 \, as \cos \phi)$$

The normal to the rod surface at P_1 is

$$(-\varepsilon s'(z), \, 1, \, 0)/(1 + (\varepsilon s'(z))^2)^{1/2}$$

and that to the furnace surface at P_1 is

$$(0, \, -\cos \phi, \, -\sin \phi)$$

The unit vector in the direction $P_1 P_2$ is

$$(\zeta - z, \, \varepsilon(a \cos \phi - s), \, \varepsilon \, a \, \sin \phi)/r_{12}$$

Therefore

$$\cos \phi_1 = \frac{\varepsilon \, ((z-\zeta)s'(z) + a \cos\phi - s(z))}{r_{12} \, \sqrt{1 + (\varepsilon s'(z))^2}}$$

$$\cos \phi_2 = \frac{\varepsilon}{r_{12}} \, (a - s(z) \cos \phi).$$

The quantity for which the limit is taken in (29) is first transformed by replacing ζ by $z - \zeta$, and for convenience set

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$$\tilde{r} = (a^2 + s(z)^2 - 2as(z) \cos\phi)^{1/2}$$

which gives

$$\frac{k_3 \, a \, \epsilon^3}{\pi \, (1+(\epsilon s')^2)^{1/2}} \int_{-\infty}^{\infty} \int_{-\phi_0}^{\phi_0} \frac{F(z, z-\zeta)(\zeta s'+a \cos\phi-s)(a-s \cos\phi)}{(\zeta^2 + \epsilon^2 \, \tilde{r}^2)^2} \, d\phi \, d\zeta$$

where s and s' are evaluated at z.

Next make the substitution

$$\zeta = \xi \, \epsilon \, \tilde{r},$$

the above quantity is then

$$\frac{k_3 \, a}{\pi(1+(\epsilon s')^2)^{1/2}} \int_{-\infty}^{\infty} \int_{-\phi_0}^{\phi_0} \frac{F(z, z-\epsilon\tilde{r}\xi)(\xi\xi s'\tilde{r}+a \cos\phi-s)(a-s \cos\phi)}{\tilde{r}^3 \, (\xi^2+1)^2} \, d\phi \, d\xi \, .$$

Now the limit can be taken easily giving

$$q_0 = \frac{k_3}{\pi} F_0(z,z) \int_{-\infty}^{\infty} \frac{d\xi}{(\xi^2+1)^2} \int_{-\phi_0(z,z)}^{\phi_0(z,z)} \frac{(a \cos\phi-s_0)(a-s_0 \cos\phi)}{(a^2+s_0^2-2as_0 \cos\phi)^{3/2}} \, a \, d\phi \, .$$

The first integral is evaluated easily as

$$\int_{-\infty}^{\infty} \frac{d\xi}{(\xi^2+1)^2} = \pi/2$$

using for example $\xi = \tan\alpha$. The second integral is also easily evaluated as follows.

First note that $\phi_0(z,z)$ is given by

$$a \cos\phi_0 = s_0(z)$$

also setting $\beta = s_0(z)/a$ the integral becomes

$$2 \int_0^{\phi_o} \frac{(\cos\phi - \beta)(1 - \beta\cos\phi)}{(1 + \beta^2 - 2\beta\cos\phi)^{3/2}} \, d\phi = \frac{2 \sin\phi}{(1 + \beta^2 - 2\beta\cos\phi)^{1/2}} \Bigg|_0^{\phi_o}$$

$$= \frac{2 \sin\phi_o}{(1 + \beta^2 - 2\beta\cos\phi_o)^{1/2}} = \frac{2\sqrt{1 - \beta^2}}{(1 + \beta^2 - 2\beta^2)^{1/2}} = 2 .$$

Therefore

$$(30) \qquad q_o = k_3 \, F_o \, (z,z)$$

$$= k_3 \, (\theta_o^4(s_o(z),z) - \theta_f^4(z))$$

Using the transformation (21) the boundary condition (27) is

$$(31) \quad \frac{\partial}{\partial\eta}(k_1\theta_o + k_2\theta_o^4) = -s_o(z)k_3(\theta_o^4(1,z) - \theta_f^4(z)).$$

Equation (27) with boundary condition (31) is again a parabolic initial boundary value problem for the temperature θ_o in terms of the coordinates (η,z) with z being the time-like variable. The temperature depends on the fluid flow through the shape function $s_o(z)$ which appears in (31) and is determined by either (22) or (24) which depend on temperature through the viscosity. The validity of the use of an average viscosity $\bar{\mu}(z)$ should be apparent from the temperature distribution of the rod. If there are significant variations in temperature in the rod when the surface begins to become liquid then the use of equation (24) would be more appropriate than (22).

An interesting feature of the formula (30) is that to within $O(\varepsilon)$ the flux is independent of the furnace radius a. It is consistent with one's intuition, however, that the effective furnace temperature is approximately that of the furnace closest to the rod. However, one might expect that (30) would be a less accurate approximation than it is.

7. Summary.

This section serves as a summary of the previous analysis. The primary problem is to determine the temperature distribution $\theta_o(r,z)$. The governing equation is (27)

$$\frac{\partial\theta_o}{\partial z} = \frac{1}{\eta} \frac{\partial}{\partial\eta} \left(\eta \frac{\partial}{\partial\eta} (k_1\theta_o + k_2\theta_o^4) \right)$$

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$$\text{for} \quad -\infty < z < \infty, \quad 0 < \eta < 1$$

with the boundary condition (31) on $\eta = 1$,

$$\frac{\partial}{\partial \eta} (k_1 \theta_o + k_2 \theta_o^4) = -s_o(z) k_3 (\theta_o^4(1,z) - \theta_f^4(z)).$$

The surface shape function $s_o(z)$ is given by (24).

$$\frac{s_o'(z)}{s_o(z)(1-(\alpha_o(z)s_o(z))^2)} = -\frac{1}{12} \left(\int_{\alpha_o(z)}^1 \mu(\theta_o(\eta,z)) \, \eta d\eta \right)^{-1}$$

where $\alpha_o(z)$ is that value of η for which

$$\theta_o < \theta^* \quad \text{for} \quad \eta < \alpha_o(z)$$

$$\theta_o > \theta^* \quad \text{for} \quad \eta > \alpha_o(z).$$

If one uses an approximate viscosity $\bar{\mu}(z)$ then (23) holds

$$s_o(z) = \exp(-\frac{1}{6} \int_{-\infty}^{z} \bar{\mu}(\zeta)^{-1} d\zeta).$$

In the numerical computation of these problems, one would take the furnace temperature and

initial rod temperature as equal, for, say, $z < 0$. i.e.

$$\theta_o(\eta,z) = \theta_f(z) = 1 \quad \text{for} \quad z < 0.$$

Simultaneously, the dopant concentration can be computed by equation (25) with boundary

condition (26)

$$\frac{\partial c_o}{\partial z} = \frac{1}{\eta} \frac{\partial}{\partial \eta} (\eta \, k_o(\theta_o(\eta,z)) \frac{\partial c_o}{\partial \eta}), \quad 0 < \eta < 1$$

$$\frac{\partial c_o}{\partial \eta} = 0 \quad \text{at} \quad \eta = 1.$$

It is also of interest to obtain expressions for the drawing radius ratio in terms of

the dimensional parameters. If a global average viscosity $\bar{\mu}^*$ is defined by

$$(\bar{\mu}^*)^{-1} = \frac{1}{Z_o} \int_{-\infty}^{\infty} \bar{\mu}(T(z))^{-1} dz$$

then (23) yields

(32) $\quad \dfrac{R_1}{R_o} = \exp(-P_d Z_o/(U_o R_o^2 \mu^*)).$

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This formula clearly show the relationship that holds between the drawing force F_d, the feed velocity U_o, the initial radius R_o, the furnace length Z_o, the average viscosity $\bar{\mu}^*$, and the radius ratio R_1/R_o. Although $\bar{\mu}^*$ depends on F_d, U_o, and R_o, presumably it is most strongly dependent on the furnace temperature distribution. Assuming this, formula (32) can be used to estimate the change in R_1 which would result from small changes in F_d, R_o, U_o, and the furnace temperature.

The advantage of the perturbation analysis is that it gives such simple expressions as (32) between the large number of parameters which govern this problem. The relation (32) is only true to within terms of order R_o/Z_o, but the experimentally obtained values for the coefficients and functions which have been assumed as known may well have errors of a similar magnitude.

REFERENCES

Geer, J. F. 19771 Slender streams with gravity: outer expansion I. Phys. Fluids 20, 1613-1621.

Geer, J. F. 1977b Slender streams with gravity: outer expansion II. Phys. Fluids 20, 1622-1630.

Geer, J. F. and Strikwerda, J. C. 1980 Vertical slender jets, J. Fluid Mech., 101, 53-63.

Geer, J. F. and Strikwerda, J. C. 1983 Vertical slender jets with surface tension, J. Fluid Mech. 135, 155-169.

Geyling, F. T. and Homsy, G. M. 1980 Extensional instabilities of the glass fibre drawing process. Glass Tech. 21, 95-62.

Manfre, G. 1969 Forces acting in continuous drawing of glass fibres, Glass Tech. 10, 99-106.

Napolitano, A. and Hawkins, E. G. (1964) Viscosity of a standard soda-lime-silica glass, J. Res. Nat. Bur. Stand, A., 68, 439-448.

Paek, U. C. and Runk, R. B. 1978 Physical behavior of the neck-down region during furnace drawing of silica fibers. J. Appl. Phys. 49, 4417-4422.

Sparrow, E. M. and Cess, R. D. 1967 Radiation Heat Transfer, Brooks/Cole Publ. Co. Belmont, Ca.

Strikwerda, J. C. and Geer, J. F. 1980 A numerical method for computing the shape of a vertical slender jet, SIAM J. Sci. Statist. Comput. 1, 449-466.

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Optical fibers are produced by heating a silica rod and pulling on the
rod to produce a thin fiber. This process involves several interacting
effects, especially the temperature dependent viscosity of the rod, and the
shape of the rod in the transition zone. In this paper the equations
describing this process are studied by means of a perturbation analysis using
as the perturbation parameter the ratio of the rod to the length of the
furnace. The silica rod is usually doped so that it will have special optical
properties and it is important to know the concentration of the dopant in the

ABSTRACT   (Continued)

final fiber. Simple equations for the significant terms in the perturbation
series are obtained for all quantities of interest. A simple formula is
obtained which expresses the radius of the fiber in terms of the initial
radius of the rod, the drawing force and other important parameters.